Models of Computation

Tutorial Exercises 2

1. Use the procedure given in the lectures to convert the following NFA

$$N = (\{ q_0, q_1, q_2 \}, \{ a, b \}, \delta, \{ q_0 \}, \{ q_0 \})$$

to a regular expression where

δ	a	b
q_0	$\{ q_0 \}$	$\{ q_1 \}$
q_1	$\{ q_2 \}$	$\emptyset$
q_2	$\{ q_0 \}$	$\{ q_1 \}$

2. (i) Prove that the following

(14) $$F\,E \leq F \;\Rightarrow\; F\,E^* \leq F$$

is a rule that is derivable from Kozen's system.

(ii) Prove each of the following equivalences between regular expressions using Kozen's system (of axioms and rules).

(a) $E^*\,E^* \equiv E^*$

(b) $(E^*)^* \equiv E^*$

(c) $(E^*\,F^*)^* \equiv (E+F)^*$

3. Prove or disprove each of the following:

(i) $(E+F)^* \equiv E^* + F^*$

(ii) $(EF+E)^*E \equiv E(FE+E)^*$

(iii) $E(FE+E)^*F \equiv EE^*F(EE^*F)^*$

4. Prove that if L is a regular language, so is $L^R = \{ w \mid \text{the reversal of } w \text{ is in } L \}$.

5. Show that the regular languages are closed under the following operations:

(i) $min(L) = \{ w \mid w \text{ is in } L \text{ but no proper prefix of } w \text{ is in } L \}$.

(ii) $max(L) = \{ w \mid w \text{ is in } L \text{ and for no } x \text{ other than } \epsilon \text{ is } wx \text{ in } L \}$

(iii) $init(L) = \{ w \mid \text{for some } x,\ wx \text{ is in } L \}$.

[*Hint*: Start with a DFA for L and perform an appropriate construction.]

6. Use the Pumping Lemma to prove that the following languages over the alphabet $\Sigma = \{ 0, 1 \}$ are not regular:

(i) $L_1 = \{ 0^n 1^m 0^n : m, n \geq 0 \}$.

(ii) $L_2 = \Sigma^* \setminus \{ 0^n 1^n : n \geq 0 \}$.

(iii) $L_3 = \{ 0^m 1^n : m \neq n \}$.

7. **Replace this question by the following two (7a and 7b)**
Consider $L = \{ w : w \text{ is not a palindrome} \}$. A *palindrome* is a string that reads the same forward or backward.

(i) Show that L satisfies the three conditions of the Pumping Lemma.

(ii) Prove that L is not regular.

(iii) Does (ii) contradict (i)?

7a Consider $L = \{ w \in \{0,1\} : w \text{ is not a palindrome} \}$. A *palindrome* is a string that reads the same forward or backward. Show that L is not regular in three different ways:

(i) using the fact that the set of palindroms is not regular;

(ii) using the Myhill-Nerode Theorem;

(iii) using the Pumping Lemma.

7b Let $L_1 = \{ 01^i01^j01^j \mid i,j > 0 \}$ and $L_2 = \{ w \in \{0,1\}^* \mid w \text{ contains } 00 \}$.

(i) Show that $L = L_1 \cup L_2$ satisfies the Pumping Lemma with pumping length 3.

(ii) Prove that $L = L_1 \cup L_2$ is not regular using the Myhill-Nerode theorem.

(iii) Does (ii) contradict (i)?

8. Fix an alphabet. Let x and y be strings and let L be any language. We say that x and y are *L-indistinguishable*, written $x \equiv_L y$, if for every string z, $xz \in L$ iff $yz \in L$. Show that $\equiv_L$ is an equivalence relation. We define the *index* of L to be the number of equivalence classes of L. (The index of L may be finite or infinite.)

(i) Show that if L is recognized by a DFA with k states, L has index at most k.

(ii) Show that if the index of L is a finite number k, it is recognized by a DFA with k states.

Hence prove the *Myhill-Nerode Theorem*: L is regular iff $\equiv_L$ has finite index; moreover the index is the size (= number of states) of the smallest DFA recognizing it.

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