

Automata, Logic and Games: An Annotated Reading List

C.-H. L. Ong

April 5, 2004

Contents

1	Infinite Automata and MSO Logic	1
1.1	Preliminaries	1
1.2	Monadic Second-Order (MSO) Logic	2
1.3	Büchi automata	2
1.4	Questions	2
2	Linear Temporal Logic	3
2.1	Alternating automata	3
2.2	Linear-time Propositional Temporal Logic	3
2.3	Questions	3
3	Modal Mu-Calculus: Syntax and Semantics	4
3.1	Questions	5
4	Modal Mu-Calculus: Fundamental Theorem	6
4.1	Pre-models and well-founded maximal paths	6
4.2	Signatures.	7
4.3	Questions	8
5	Modal Mu-Calculus: Model Checking and Parity Games	9

1 Infinite Automata and MSO Logic

1.1 Preliminaries

Readings: Standard texts e.g. [Sip97], [Var95, §1-2].

Automata on finite words. Regular languages.

1.2 Monadic Second-Order (MSO) Logic

Readings: Relevant parts of [Tho97, p. 1-8].

Fix a finite alphabet Σ . Word model for finite and infinite words. Relational structure

$$\underline{w} = \langle \text{dom}(w), S^w, <^w, (Q_a^w)_{a \in \Sigma} \rangle$$

Monadic Second-Order (MSO) Logic (over words). Prenex normal form. MSO_0 -logic has atomic formulas:

$$X \subseteq Y, \text{Sing}(X), \text{Succ}(X, Y), X \subseteq Q_a \quad (a \in \Sigma)$$

and no first-order variables. MSO and MSO_0 are equally expressive. The language (resp. ω -language) defined by an MSO sentence ϕ , $L(\phi)$ (resp. $L_\omega(\phi)$).

Exercise. Write down MSO sentences defining the following languages:

- (i) The set of words in which after each a , there is a b .
- (ii) The set of words in which a appears only at odd, or only at even positions.

1.3 Büchi automata

Readings: Relevant parts of [Var95, §1-2] and [Tho97, §5].

Automata on infinite words. Non-deterministic and deterministic Büchi automata.

Other acceptance conditions: Muller, Rabin. Equivalence of non-deterministic Muller, Rabin and Büchi automata. ω -languages acceptable by deterministic Muller automata are closed under complementation.

Automata constructions: union, intersection, complementation (Büchi 62, proof not required).

Determinization. McNaughton's Theorem (1966): Every Büchi automaton can be effectively transformed into an equivalent deterministic Muller automaton. Safra's Theorem (1988): Every Büchi automaton with n states can be effectively transformed into an equivalent Rabin automaton with $2^{O(n \log n)}$ states and n pairs in the acceptance conditions. [Proofs not required.]

An ω -language is acceptable by a Büchi automaton iff it is MSO-definable, and the transformations between Büchi automata and MSO-formulas are effective. Büchi's Theorem: The theory $S1S$ (i.e. MSO-sentences true in the structure $(\omega, S, <)$) is decidable.

Decision problems for Büchi automata: Nonemptiness and Nonuniversality.

1.4 Questions

1. Show that *deterministic* Büchi automata are not closed under complementation. I.e. there is a deterministic Büchi-acceptable ω -language whose complement is not deterministic Büchi-acceptable.
2. Prove that an ω -language is acceptable by a Büchi automaton iff it is MSO-definable.

2 Linear Temporal Logic

2.1 Alternating automata

Readings: [Var95].

Alternating Büchi automata. Non-deterministic Büchi automata and alternating Büchi automata are equally expressive.

2.2 Linear-time Propositional Temporal Logic

Readings: [Var95].

Formulas of *LTL*. Interpretation of *LTL* formulas over computations: $\pi, i \models \phi$, where $\pi : \omega \rightarrow 2^{Prop}$.

Theorem. Given an *LTL* formula ϕ , there is an alternating Büchi automaton $A_\phi = (\Sigma, S, s^0, \rho, F)$ where $\Sigma = 2^{Prop}$ and $|S| = O(|\phi|)$, such that $L_\omega(A_\phi)$ is exactly the set of computations satisfying the formula ϕ .

2.3 Questions

1. Set $\Sigma = \{0, 1\}$.

$$\begin{aligned} L_1 &= \{ \alpha \in \Sigma^\omega : \alpha \text{ contains finitely many occurrences of } 1 \} \\ L_2 &= \{ \alpha \in \Sigma^\omega : 0 \text{ occurs in all even positions of } \alpha \} \end{aligned}$$

Construct a Büchi automaton and an MSO_0 -sentence that define each of the above ω -languages.

2. Prove the Theorem: $L \subseteq A^\omega$ is recognized by a Büchi automaton iff L is a finite union of sets of the form JK^ω , where $J \subseteq A^*$, $K \subseteq A^+$ are regular languages.

For this reason, such ω -languages are called *ω -regular*.

3. Construct the alternating Büchi automaton associated with the *LTL*-formula

$$pU(X\neg q).$$

4. We define a sublogic $\mathcal{T}(U)$ of *LTL* consisting of formulas that are built up from the atomic propositions, using conjunction, negation and the until-operator $\phi U \psi$. (Thus we may write *LTL* as $\mathcal{T}(X, U)$.)

Set $Prop = \{q\}$. Consider the computation

$$r = \{q\} \{q\} \{ \} \{ \} \{ \} \{ \} \cdots$$

meaning that $r(0) = r(1) = \{q\}$ and $r(i) = \{ \}$ for $i \geq 2$.

- (i) Prove, by induction on the complexity of formulas, that for all $\phi \in \mathcal{T}(U)$, $r, 0 \models \phi$ iff $r, 1 \models \phi$.

(ii) Find an *LTL*-formula ψ satisfying $r, 0 \models \psi$ and $r, 1 \not\models \psi$.

Hence prove that $\mathcal{T}(U)$ is strictly less expressive than *LTL*

5. We define a sublogic $\mathcal{T}(X, G)$ of *LTL* consisting of formulas that are built up from the atomic propositions, using conjunction, negation, next-time operator $X\phi$, and the always-modality $G\phi$.

We shall prove: *Theorem.* $\mathcal{T}(X, G)$ is strictly less expressive than *LTL*.

Consider a model that has states s_0, \dots, s_{4m-1} and the transition relation is specified by:

$$s_i \longrightarrow s_j \quad \leftrightarrow \quad j = i + 1, \text{ or } i = 4m - 1 \text{ and } j = 2m.$$

The proposition q_1 is assumed to hold for all states except s_{3m} , and q_2 is assumed to hold for just the states s_{2m-1} and s_{4m-1} . Take $Prop = \{q_1, q_2\}$, and let r be the uniquely determined computation (or run) starting in state s_0 . I.e.

$$r = \underbrace{\{q_1\} \cdots \{q_1\}}_{2m} \{q_1, q_2\} \left(\underbrace{\underbrace{\{q_1\} \cdots \{q_1\}}_m \{ \} \{q_1\} \cdots \{q_1\}}_{2m} \{q_1, q_2\} \right)^\omega$$

- (i) List the elements of the set $\{0 \leq l \leq 4m - 1 : r, l \models q_1 U q_2\}$.
(ii) Prove that for all $\phi \in \mathcal{T}(X, G)$ containing fewer than $m - 1$ occurrences of X , we have

$$r, 0 \models \phi \quad \leftrightarrow \quad r, 2m \models \phi.$$

Observe that $r, 0 \models q_1 U q_2$ and $r, 2m \not\models q_1 U q_2$. Hence or otherwise prove that $\mathcal{T}(X, G)$ is strictly less expressive than *LTL*.

3 Modal Mu-Calculus: Syntax and Semantics

*Readings*¹: [BS01, §3], and relevant parts of [Sti97].

Modal mu-calculus formulas, with respect to

- Var , a set of variable names ranged over by X, Y, Z , etc.
- $Prop$, a set of atomic propositions
- \mathcal{L} , a set of labels.

Positive, and positive normal forms.

Modal mu-calculus structures over $(Prop, \mathcal{L})$: A *labelled transition system* (or *Kripke structure*) is a triple $T = \langle S, \rightarrow, \rho \rangle$ where

¹The papers can be downloaded from <http://www.dcs.ed.ac.uk/home/cps/>.

- S is a set of states
- a transition relation $\rightarrow \subseteq S \times \mathcal{L} \times S$ (as usual we write $s \xrightarrow{a} t$)
- a function $\rho : Prop \rightarrow \mathcal{P}(S)$ mapping each atomic proposition to a set of states.

Definition of $\|\phi\|_V^T$, the set of states in the labelled transition system $T = \langle S, \rightarrow, \rho \rangle$ that satisfy ϕ w.r.t. $V : Var \rightarrow \mathcal{P}(S)$, a valuation function.

Approximants: Definition of $\|\mu Z^\alpha.\phi\|_V^T$ and $\|\nu Z^\alpha.\phi\|_V^T$ where α ranges over ordinals.

3.1 Questions

1. Prove Tarski's Theorem: Let $\langle L, \leq \rangle$ be a complete lattice. If $f : L \rightarrow L$ is a monotone function, then the greatest νf and least μf fixpoints of f exist, and are given by:

$$\begin{aligned}\mu f &= \bigwedge \{x \in L : f(x) \leq x\} \\ \nu f &= \bigvee \{x \in L : x \leq f(x)\}.\end{aligned}$$

Show that $\langle \mathcal{P}(S), \subseteq \rangle$ is a complete lattice. Hence justify the following:

$$\begin{aligned}u \in \|\mu Z.\phi\|_V^T &\leftrightarrow u \in \mu(f_{\phi,Z}) \\ u \in \|\nu Z.\phi\|_V^T &\leftrightarrow u \in \nu(f_{\phi,Z})\end{aligned}$$

defining the function $f_{\phi,Z}$.

2. Prove the Proposition:

$$\begin{aligned}u \in \|\mu Z.\phi\|_V^T &\leftrightarrow u \in \|\mu Z^\alpha.\phi\|_V^T \text{ for some } \alpha \\ u \in \|\nu Z.\phi\|_V^T &\leftrightarrow u \in \|\nu Z^\alpha.\phi\|_V^T \text{ for all } \alpha\end{aligned}$$

3. We say that ϕ and ψ are *equivalent* just if for any T and for any V , we have $\|\phi\|_V^T = \|\psi\|_V^T$.

Prove

- (i) $\mu Z.\phi(Z)$ and $\phi(\mu Z.\phi)$ are equivalent (similarly $\nu Z.\phi$ and $\phi(\nu Z.\phi)$ are equivalent).
- (ii) $\mu Z.\phi(Z)$ and $\neg \nu Z.\neg \phi(\neg Z)$ are equivalent.

4. (*Hard*) We say that variable Z in a formula ϕ is *guarded* if every occurrence of Z in ϕ is in the scope of some modal operator $[a]$ - or $\langle a \rangle$ -. We say that a formula is *guarded* if for every subformula $\sigma Z.\psi$ of ϕ , Z is guarded in ψ (where $\sigma = \mu, \nu$).

Prove Kozen's Lemma: Every modal mu-calculus formula is equivalent to some positive guarded formula.

5. Is there a labelled transition system T and a valuation V such that for every modal mu-calculus formula ϕ , $\|\mu Z.\phi\|_V^T = \|\nu Z.\phi\|_V^T$?

6. Show that $s_0 \in \|\mu Z.[a]Z\|_V^T$ iff there is no infinite transition sequence

$$s_0 \xrightarrow{a} s_1 \xrightarrow{a} s_2 \xrightarrow{a} \dots$$

What is the meaning of $\nu Z.[a]Z$?

4 Modal Mu-Calculus: Fundamental Theorem

Readings: [BS01, §3].

Our goal is to prove:

Fundamental Theorem. *Let $T = \langle S, \rightarrow, \rho \rangle$ be a labelled transition system. A well-founded pre-model (of T for ϕ w.r.t. V) is a model i.e. if (s, ϕ) is in the pre-model then $s \in \|\phi\|_V^T$. \square*

and also its converse.

Let ϕ be a modal mu-formula. We write $\mathbf{Sub}(\phi)$ for the set of all subformulas of ϕ . Suppose ϕ is in normal form (i.e. all variables are renamed appropriately to be distinct), and $\sigma_1 Z_1.\phi_1, \sigma_2 Z_2.\phi_2 \in \mathbf{Sub}(\phi)$. We say that Z_1 *subsumes* Z_2 if $\sigma_2 Z_2.\phi_2 \in \mathbf{Sub}(\sigma_1 Z_1.\phi_1)$.

In the following we fix

- a labelled transition system $T = \langle S, \rightarrow, \rho \rangle$
- a modal mu-calculus formula ϕ , and
- a valuation $V : Var \rightarrow \mathcal{P}(S)$.

4.1 Pre-models and well-founded maximal paths

A *pre-model*² of T for ϕ w.r.t. V is a directed graph

$$\langle M \subseteq S \times \mathbf{Sub}(\phi), \succ \subseteq M \times M \rangle$$

satisfying the following *local consistency* conditions: (we read $\theta \succ \theta'$ as “ θ depends on θ' ”)

1. If $(s, \phi_1 \wedge \phi_2) \in M$ then $(s, \phi_1 \wedge \phi_2) \succ (s, \phi_i)$, for $i = 1$ and 2 .
2. If $(s, \phi_1 \vee \phi_2) \in M$ then $(s, \phi_1 \vee \phi_2) \succ (s, \phi_i)$, for a *chosen* i i.e. $(s, \phi_1 \vee \phi_2)$ has exactly one successor.
3. If $(s, [a]\phi) \in M$ then for all t such that $s \xrightarrow{a} t$, $(s, [a]\phi) \succ (t, \phi)$.
4. If $(s, \langle a \rangle \phi) \in M$ then

²Our definition combines a quasi-model in the sense of [BS01] with a *choice function* that selects an outgoing edge from vertices of the forms $(s, \phi_1 \vee \phi_2)$ and $(s, \langle a \rangle \phi)$ (if appropriate).

if s has an a -successor in T then $(s, \langle a \rangle \phi) \succ (t, \phi)$ for a *chosen* (i.e. exactly one) t such that $s \xrightarrow{a} t$, otherwise $(s, \langle a \rangle \phi)$ has no outgoing edge.

5. If $(s, \sigma Z.\phi) \in M$ then $(s, \sigma Z.\phi) \succ (s, Z)$.

6. If $(s, Z) \in M$ and Z is bound to the fixpoint formula $\sigma Z.\phi$ then $(s, Z) \succ (s, \phi)$.

and such that every maximal path in the graph that is finite ends in one of the following vertices:

- (s, t)
- (s, p) and $s \in \rho(p)$
- $(s, \neg p)$ and $s \notin \rho(p)$
- (s, X) and $s \in V(X)$, and X is free in ϕ
- $(s, \neg X)$ and $s \notin V(X)$, and X is free in ϕ
- $(s, [a]\psi)$.

A maximal path in a pre-model is said to be *well-founded* if it is finite, or if the largest (w.r.t. subsumption) variable that occurs infinitely often in it is a ν -variable. A pre-model (of T for ϕ at s_0 w.r.t. V) is said to be *well-founded* if every maximal path is well-founded.

4.2 Signatures.

A signature is just a sequence of ordinals. We say that signatures $r < r'$ if r lexicographically precedes r' .

Let ϕ be a formula in normal form. Let

$$\sigma_1 Z_1.\psi_1, \sigma_2 Z_2.\psi_2, \dots, \sigma_n Z_n.\psi_n$$

be the set of fixpoint formulas in ϕ , in decreasing order of size. So if $i < j$ it is impossible for Z_j to subsume Z_i . Further let

$$\mu Y_1.\chi_1, \mu Y_2.\chi_2, \dots, \mu Y_m.\chi_m$$

be the set of least fixpoint formulas in ϕ , again in decreasing order of size.

Given a signature $r = \alpha_1, \dots, \alpha_m$ and a valuation V , we define valuations V_0^r, \dots, V_n^r by induction:

$$\begin{aligned} V_0^r &= V \\ V_{i+1}^r &= V_i^r[E_{i+1}/Z_{i+1}] \end{aligned}$$

where

$$E_{i+1} = \begin{cases} \|\sigma_{i+1} Z_{i+1}.\psi_{i+1}\|_{V_i^r}^T & \text{if } \sigma_{i+1} = \nu \\ \|\mu Y_j^{\alpha_j}.\chi_j\|_{V_i^r}^T & \text{else if } \sigma_{i+1} = \mu Y_j \end{cases}$$

Thus we use a signature $r = \alpha_1, \dots, \alpha_m$ to interpret the least fixpoint formulas in ϕ so that the j -th such is interpreted by its α_j -th approximant.

4.3 Questions

1. Take a pre-model such that each vertex (s, ψ) is given a signature $r = \alpha_1, \dots, \alpha_m$. We say that the signature assignment is *locally consistent* provided:
 - (a) If $\phi = \phi_1 \wedge \phi_2$ then (s, ϕ_1) and (s, ϕ_2) have signatures $\leq r$.
 - (b) If $\phi = \phi_1 \vee \phi_2$ then for some i , (s, ϕ_i) has signature $\leq r$.
 - (c) If $\phi = [a]\chi$ then for all t such that $s \xrightarrow{a} t$, (t, χ) has signature $\leq r$.
 - (d) If $\phi = \langle a \rangle \chi$ then there exists t such that $s \xrightarrow{a} t$ and (t, χ) has signature $\leq r$.
 - (e) If $\phi = \nu Z.\psi$ then (s, Z) has signature r . Similarly if $\phi = Z$ and Z is bound to $\nu Z.\psi$ then (s, ψ) has signature r .
 - (f) If $\phi = \mu Y_j.\chi_j$ then (s, Y_j) has signature r' such that $r' =_{j-1} r$, meaning that both have the same first $j - 1$ entries. And if $\phi = Y_j$ then (s, χ_j) has signature that strictly decreases in the j -th component, but unchanged in the first $j - 1$ components.

Prove the *Lemma*. If a pre-model (of T for ϕ w.r.t. V) has a locally consistent signature assignment, then for each (s, ψ) in the pre-model with assigned signature r , we have $s \in \|\psi\|_{V_r}^T$.

2. Suppose $\langle M, \succ \rangle$ is a well-founded pre-model of T for ϕ w.r.t. V , and assume the notations for fixpoints in the preceding.

Signature assignment. We assign an m -long signature a_1, \dots, a_m to each vertex (s, ψ) of the pre-model so that a_j is the maximum number of occurrences of Y_j along any path from (s, ψ) without meeting a larger (w.r.t. subsumption) variable.

Prove that the signature assignment is locally consistent.

Hence deduce the Fundamental Theorem.

3. We aim to prove the converse of the Fundamental Theorem:

Let ϕ be a modal mu-calculus formula. Suppose $s \in \|\phi\|_V^T$, for some T, V and s . Then T has a pre-model for ϕ that is well-founded.

Construction of the pre-model. Take T . Annotate each state s by the subformulas of ϕ that it satisfies w.r.t. V . For each such pair (s, ψ) , assign the $<$ -least signature r such that $s \in \|\psi\|_{V_r}^T$, and always select the \succ -successor to be that with the least signature in the cases of $(s, \phi_1 \vee \phi_2)$ and $(s, \langle a \rangle \chi)$.

Formalize the construction and prove that it is a well-founded pre-model.

Answers

3. To show that the pre-model is well-founded, take an infinite \succ -path $(s_0, \phi_0), (s_1, \phi_1), \dots$. Suppose Y , bound to $\sigma Y.\psi$, is the largest variable that occurs infinitely often in the path. Let (s_k, ϕ_k) be the node in the path after which every occurrence of a fixpoint variable is subsumed by Y . Let k_1, k_2, \dots be the positions in the path where Y occurs from that point onwards. Suppose, for a contradiction, Y is a μ -variable, say, $Y = Y_j$ and bound to $\mu Y_j.\chi_j$.

Consider the \succ -path from k_i to k_{i+1} . The dependency (\succ -successor) for (s_k, Y) is (s_k, χ_j) , which causes a decrease in the j -th position of the signature. We show that the remaining nodes in the path from k_i to k_{i+1} cannot cancel this decrease. Clearly, the only case we need worry about is the case of $(s, \mu Z.\psi) \succ (s, \psi)$, as this may cause an increase in signature. By assumption, Z is subsumed by Y . It follows that $Z = Y_l$ for some $l > j$. Hence the increase in the signature occurs after the j -th position. Thus there is still an overall decrease from k_i to k_{i+1} . Thus there is a infinite sequence of strictly decreasing signatures, which gives a contradiction.

5 Modal Mu-Calculus: Model Checking and Parity Games

Aim. We characterize the satisfaction relation $s \models_V^T \phi$ (meaning $s \in \|\phi\|_V^T$) between a state s and a modal mu-calculus formula ϕ in terms of game playing. We then consider some consequences for the finite state case.

Reading: [Sti97, §4].

Notation. In [Sti97], $E \models_V^T \phi$ means $E \in \|\phi\|_V^T$ for some labelled transition system. Players I and II are sometimes called R (for Refuter) and V (for Verifier) respectively.

Fix a labelled transition system $T = \langle S, \rightarrow, \rho \rangle$ and a valuation V . Take a modal mu-calculus formula in normal form ϕ .

Questions

1. Prove the Proposition: If $s \models_V^T \phi$ then V (or Player II) has a history-free winning strategy.

[Hint: Similar to the converse of the Fundamental Theorem.]

2. Question 2 We aim to prove the converse of the Proposition.

Let ϕ be a formula in normal form. Let

$$\sigma_1 Z_1.\psi_1, \sigma_2 Z_2.\psi_2, \dots, \sigma_n Z_n.\psi_n$$

be the set of fixpoint formulas in ϕ , in decreasing order of size. So

- (i) if $i < j$ it is impossible for Z_j to subsume Z_i ;
- (ii) if Z_i subsumes Z_j then $i \leq j$.

Further let

$$\nu Y_1 \cdot \chi_1, \nu Y_2 \cdot \chi_2, \dots, \nu Y_m \cdot \chi_m$$

be the set of greatest fixpoint formulas in ϕ , again in decreasing order of size.

We define valuations V_0, \dots, V_n by induction:

$$\begin{aligned} V_0 &= V \\ V_{i+1} &= V_i[\|\sigma_{i+1} Z_{i+1} \cdot \psi_{i+1}\|_{V_i}^T / Z_{i+1}] \end{aligned}$$

Thus we can make sense of $\|\psi\|_{V_n}^T \subseteq S$, for any $\psi \in \mathbf{Sub}(\phi)$.

Given a signature $r = \alpha_1, \dots, \alpha_m$ and a valuation V , we define ν -valuations³ V_0^r, \dots, V_n^r by induction:

$$\begin{aligned} V_0^r &= V \\ V_{i+1}^r &= V_i^r[E_{i+1}/Z_{i+1}] \end{aligned}$$

where

$$E_{i+1} = \begin{cases} \|\sigma_{i+1} Z_{i+1} \cdot \psi_{i+1}\|_{V_i}^T & \text{if } \sigma_{i+1} = \mu \\ \|\nu Y_j^{\alpha_j} \cdot \chi_j\|_{V_i^r}^T & \text{if } \sigma_{i+1} = \nu Y_j \end{cases}$$

Recall that if $s \notin \|\nu Z \cdot \psi\|_V^T$ then there is an ordinal α such that $s \notin \|\nu Z^\alpha \cdot \psi\|_V^T$ and for all $\beta < \alpha$, we have $s \in \|\nu Z^\beta \cdot \psi\|_V^T$. Can you prove this?

Let $\psi \in \mathbf{Sub}(\phi)$. If $s \notin \|\psi\|_V^T$, we define the ν -signature of (s, ψ) , written $\text{sig}^\nu(s, \psi)$, to be the $<$ -least signature $r = \alpha_1, \dots, \alpha_m$ such that $s \notin \|\psi\|_{V_n^r}^T$.

Prove the *Signature Decrease Lemma*: Whenever the left-hand-side of the equation is defined:

1. $\text{sig}^\nu(s, \phi_1 \wedge \phi_2) = \text{sig}^\nu(s, \phi_1)$ or $\text{sig}^\nu(s, \phi_1 \wedge \phi_2) = \text{sig}^\nu(s, \phi_2)$
2. $\text{sig}^\nu(s, \phi_1 \vee \phi_2) = \max(\text{sig}^\nu(s, \phi_1), \text{sig}^\nu(s, \phi_2))$
3. $\text{sig}^\nu(s, [a]\phi_1) \geq \text{sig}^\nu(t, \phi_1)$, for some t such that $s \xrightarrow{a} t$
4. $\text{sig}^\nu(s, \langle a \rangle \phi_1) \geq \text{sig}^\nu(t, \phi_1)$ for some t such that $s \xrightarrow{a} t$, or s does not have an a -transition
5. Assuming Z_i is a μ -variable, $\text{sig}^\nu(s, \mu Z_i \cdot \psi_i) = \text{sig}^\nu(s, Z_i) = \text{sig}^\nu(s, \psi_i)$
6. Assuming Z_i is a ν -variable, $\text{sig}^\nu(s, \nu Z_i \cdot \psi_i)$ is the same as $\text{sig}^\nu(s, \psi_i)$ on the first $i - 1$ components
7. Assuming Z_i is a ν -variable, $\text{sig}^\nu(s, Z_i) > \text{sig}^\nu(s, \psi_i)$, and the first $i - 1$ components of the two signatures are the same but not the i -component.

Hence prove that if $s \not\models_V^T \phi$ then R (or Player I) has a history-free winning strategy.

³This is dual to, and not to be confused with, the μ -valuations as defined in the preceding Exercise.

References

- [BS01] J. Bradfield and C. Stirling. Modal logics and mu-calculi. In J. Bergstra, A. Ponse, and S. Smolka, editors, *Handbook of Process Algebra*, pages 293–332. Elsevier, North-Holland, 2001.
- [Sip97] M. Sipser. *Introduction to the Theory of Computation*. PWS Publishing Company, 1997.
- [Sti97] C. Stirling. Bisimulation, model checking and other games. Notes for Mathfit instructional meeting on games and computation, Edinburgh, June 1997.
- [Tho97] W. Thomas. Languages, automata and logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, volume 3. Springer-Verlag, 1997.
- [Var95] M. Y. Vardi. An automata-theoretic approach to linear temporal logic. In *Banff Higher Order Workshop*, pages 238–266. Springer-Verlag, 1995.

C.-H. L. Ong
Trinity 2004