MSc (Mathematics and Foundations of Computer Science)

Automata, Logic and Games

Hilary Term 2004

A Σ-*labelled tree* T (with branching degree bounded by k) is a partial function $T : \{ 1, 2, \cdots, k \}^* \longrightarrow \Sigma$ such that the domain of definition $\mathsf{dom}(T)$ is prefix-closed. The root of the tree is denoted ϵ, and if $vi \in \mathsf{dom}(T)$ then vi is said to be the i-th successor of v. We adopt the convention that $v0 = v$. We say that the tree is *full k-ary* just in case every node of the tree has k successors; this is equivalent to saying that $\mathsf{dom}(T) = \{ 1, 2, \cdots, k \}^*$. In the following, we shall represent Σ-labelled full k-ary trees by the structure $T = \langle \{ 1, \cdots, k \}^*, (P_a^T)_{a \in \Sigma} \rangle$ where P_a^T is a subset of $\{ 1, \cdots, k \}^*$ consisting of nodes that are labelled a.

We consider *modal mu-calculus formulas* (for Σ-labelled full k-ary trees), with respect to

- $Var = \{ X, Y, Z, \cdots \}$, a set of variable names, and
- $\{ P_a : a \in \Sigma \}$, a set of propositional letters.

The syntax is defined by the grammar

$$X \mid P_a \mid \neg\phi \mid \phi \vee \psi \mid \langle\rangle\phi \mid \mu Z.\chi$$

with the usual requirement of positive occurrences of Z in χ. The greatest fixpoint $\nu Z.\chi$ and the box modality $[]\phi$ are then defined in term of the above in the standard way.

We write $[k]$ to mean $\{ 1, \cdots, k \}$. Fix a Σ-labelled full k-ary tree T. Relative to a valuation $V : Var \longrightarrow \mathcal{P}([k]^*)$, we define $\|\phi\|_V^T \subseteq [k]^*$ as follows:

$$\begin{array}{rcl}
\|X\|_V^T &=& V(X) \\
\|P_a\|_V^T &=& P_a^T \\
\|\neg\phi\|_V^T &=& [k]^* \setminus \|\phi\|_V^T \\
\|\phi_1 \vee \phi_2\|_V^T &=& \|\phi_1\|_V^T \cup \|\phi_2\|_V^T \\
\|\langle\rangle\phi\|_V^T &=& \{ v : \exists 1 \le i \le k \,.\, vi \in \|\phi\|_V^T \} \\
\|\mu Z.\phi\|_V^T &=& \bigcap \{ U \subseteq [k]^* : \|\phi\|_{V[U/Z]}^T \subseteq U \}
\end{array}$$

Alternating parity tree automata

An *alternating parity tree automaton* is a 5-tuple

$$A \;=\; \langle \Sigma, Q, \delta, q_0, \Omega \rangle$$

where Q is a set of states, $q_0 \in Q$ is the initial state,

$$\delta : Q \times \Sigma \longrightarrow \mathcal{B}^+(\{ 0, 1, \cdots, k \} \times Q)$$

is the transition function, and $\Omega : Q \longrightarrow \mathbb{N}$ assigns a numeric index to each state. By $\mathcal{B}^+(X)$ we mean the set of positive boolean formulas over X i.e. boolean formulas built from elements in X using $\wedge$ and $\vee$, where we also allow formulas true and false. For $Y \subseteq X$ and a formula $\theta \in \mathcal{B}^+(X)$, we say that Y *satisfies* θ just if assigning true to elements in Y and assigning false to elements in $X \setminus Y$ makes θ true. For example, taking X to be $\{ 0, 1, \cdots, k \} \times Q$, the sets $\{ (1, q_1), (2, q_2) \}$, $\{ (1, q_1), (1, q_3), (2, q_2) \}$ and $\{ (1, q_3), (2, q_2) \}$ satisfy $((1, q_1) \vee (1, q_3)) \wedge (2, q_2)$; every subset of X satisfies true, but no subset satisfies false.

We use such an automaton as an accepting device for Σ-labelled full k-ary trees. We formalize the notions of a play and acceptance of the automaton A in terms of a game between the two players V and R. Given a tree T, the *acceptance game* $\mathcal{G}_{A,T}$ is defined as follows:

- The initial position is the pair $(\epsilon, q_0) \in [k]^* \times Q$.
- Suppose the current position is the pair $(s, q) \in [k]^* \times Q$. Then it is V's turn to move: V picks a set $\{ (d_1, q_1), \cdots, (d_l, q_l) \}$ (say) that satisfies the boolean formula $\delta(q, T(s))$, and his move is the set $\{ (sd_1, q_1), \cdots, (sd_l, q_l) \}$. Note that by convention $s0 = s$.
- Suppose the current position is the set $\{ (s_1, q_1), \cdots, (s_l, q_l) \}$. Then it is R's turn to move: R chooses as the next move some pair (s_i, q_i) from the set.

Suppose a play terminates. Then the player that has no move to make is deemed to lose (i.e. the other player wins). If an infinite play arises

$$(s_0, q_0)\ C_1\ (s_1, q_1)\ C_2\ (s_2, q_2)\ \cdots$$

Then V wins if and only if $\min \mathsf{inf}(\Omega(q_0)\,\Omega(q_1)\,\Omega(q_2)\cdots)$ is even, where for $w \in \mathbb{N}^\omega$, we write $\mathsf{inf}(w)$ to mean the set of numbers that occur infinitely often in w.

We say that A *accepts* a tree T if V has a (history-free) winning strategy for $\mathcal{G}_{A,T}$. The *language determined by* A, written $L(A)$, is the set of trees accepted by A.

Alternating parity tree automaton determined by ϕ

Fix a modal mu-calculus sentence ϕ in positive normal form. Let

$$\sigma_1 Z_1.\psi_1,\ \sigma_2 Z_2.\psi_2,\ \cdots,\ \sigma_n Z_n.\psi_n$$

be the set of fixpoint formulas in ϕ, in decreasing order of size. We construct an alternating parity tree automaton

$$A_\phi \;=\; \langle\, \Sigma, \mathsf{Sub}(\phi), \delta_\phi, \phi, \Omega_\phi \,\rangle$$

The transition function δ_ϕ is defined by:

$$\delta_\phi \;:\; \begin{cases} (\mathsf{t}, a) & \mapsto \; \mathsf{true} \\ (\mathsf{f}, a) & \mapsto \; \mathsf{false} \\ (P_a, b) & \mapsto \; \begin{cases} \mathsf{true} & \text{if } a = b \\ \mathsf{false} & \text{otherwise} \end{cases} \\ (\phi_1 \vee \phi_2, a) & \mapsto \; (0, \phi_1) \vee (0, \phi_2) \\ (\phi_1 \wedge \phi_2, a) & \mapsto \; (0, \phi_1) \wedge (0, \phi_2) \\ (\langle\rangle\psi, a) & \mapsto \; \bigvee_{i=1}^{k} (i, \psi) \\ ([]\psi, a) & \mapsto \; \bigwedge_{i=1}^{k} (i, \psi) \\ (\sigma Z_i.\psi_i, a) & \mapsto \; (0, Z_i) \\ (Z_i, a) & \mapsto \; (0, \psi_i) \end{cases}$$

Recall that the *alternation depth* of a σ-variable Z, written $d(Z)$, is the maximum number of alternations between μ- and ν-variables in a chain $Z >_\phi X_1 >_\phi \cdots >_\phi X_k$, where $X >_\phi Y$ means "X subsumes Y". We define $\Omega_\phi : \mathsf{Sub}(\phi) \longrightarrow \mathbb{N}$ by

$$\Omega_\phi(\psi) \;=\; \begin{cases} 2 \times (D - d(Z)) & \text{if } \psi \text{ is a } \nu\text{-variable} \\ 2 \times (D - d(Z)) + 1 & \text{if } \psi \text{ is a } \mu\text{-variable} \\ N & \text{otherwise} \end{cases}$$

where D is the maximal alternation depths of σ-variables in ϕ, and N is some fixed large number.

Prove the following:

Theorem 1. *For every Σ-labelled full k-ary tree T, and for any modal mu-calculus sentence ϕ, we have $\epsilon \in \|\phi\|^T$ if and only if $T \in L(A_\phi)$.*

Hint. You may find it helpful to use μ-signatures and a Signature Decrease Lemma in one direction, and ν-signatures and a corresponding Signature Decrease Lemma in the other direction.

Optional parts

Do as much of the following as you can.

Now we revert to the modal mu-calculus for LTS (Labelled Transition Systems), i.e. relative to $(Var, Prop, \mathcal{L})$ with LTS as the corresponding structures, as defined in the reading notes.

1. Prove that the modal mu-calculus has the *Tree Model Property*: if a formula has a model, it has a model that is a tree structure (call it the *unravelled tree*) with bounded branching degree. We say that an LTS $T = \langle S, \rightarrow, \rho \rangle$ is a *tree structure* with bounded branching degree if S is a tree (i.e. a prefix-closed subset of $\{1, \cdots, k\}^*$), and whenever $s \xrightarrow{a} t$ then t is a successor of s.

2. Prove that the modal mu-calculus is decidable:

   Theorem 2. *It is decidable whether a given modal mu-calculus formula has a model.*

   You may wish to prove the Theorem by establishing: For any modal mu-calculus formula ϕ (for LTS), there is an alternating parity tree automaton $A[\phi]$ such that $L(A[\phi])$ is exactly the set of LTSs which are tree structures with bounded branching degree satisfying ϕ at the root.

   Hint. In representing the unravelled tree as a labelled full k-ary tree, you should note that the former is not necessarily a *full* k-ary tree, and the edges of the latter are not labelled. Thus you may wish to introduce the following atomic propositions: for $v \in [k]^*$

   - $P_{\text{def}}(v)$ is true iff v is part of the unravelled LTS (so that the nodes at which P_{def} is false are dummy nodes).
   - For each $b \in \mathcal{L}$, $C_b(vi)$ is true iff $v \xrightarrow{b} vi$ in the unravelled LTS.

   You may assume that the Non-Emptiness Problem for alternating parity tree automata is *EXPTIME*-decidable.

3. Prove that the modal mu-calculus has the *finite model property*:

   Theorem 3. *If a modal mu-calculus formula has a model, it has a finite model of size exponential in the size of the formula.*

4. Comment on the significance of Theorem 1. Are there other ways of proving the finite model property?

Important Note. You may assume standard results, but they should be stated precisely and in full. If you make use of any ideas (results, arguments, etc.) from the literature, you are expected to give proper citations in the standard way.